Is Rule by Majorities Special?

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Web Appendix.

Terminology and notation.

i) Policies are points in real space $\Re^p$, p ≥ 1. There are $n > 1$ voters. The declared position of voter i is regarded as a column vector, $\boldsymbol{i}$. The component of $\boldsymbol{i}$ on the m-th dimension, i_m, is i's position on m.

iii) **V** is a (p x n) declaration matrix, with n columns corresponding to the position vectors of the voters. A collective choice function (*ccf*), $\boldsymbol{d}(\ .\ )$, maps from the set of possible declarations to $\Re^p$. .

iv) $\boldsymbol{d}(\ .\ )$ is resolute if the image of any **V** is a single point.

Suppose $\boldsymbol{d}(\ .\ )$ is resolute. Then

v) If **V'** consists of a permutation of the columns of **V,** then if $\boldsymbol{d}(\mathbf{V'}) = \boldsymbol{d}(\mathbf{V})$, the *ccf* is anonymous.

vi) Let $u_i(\ .)$ be the utility function of a representative voter, i, assumed strictly quasi-concave and continuous. A voter's declared position is sincere if it corresponds to her ideal point. Let **V** be a sincere declaration and **V'** exactly the same as **V** except that voter i does not declare his ideal point. Then strategy-proofness requires that, for any such voter i, $u_i(\boldsymbol{d}(\mathbf{V})) \geq u_i(\boldsymbol{d}(\mathbf{V'}))$ for any such **V'**.

vii) A *ccf* is responsive if it exhibits:

a) Restricted responsiveness. Suppose that in declarations **V** and **V'** all voters' positions are the same except i's position changes from $\boldsymbol{i}$ to $\boldsymbol{i} + \Delta\boldsymbol{i}$. Then $\boldsymbol{d}(\mathbf{V'}) = \boldsymbol{d}(\mathbf{V}) + \mathbf{K}\Delta\boldsymbol{i}$, where **K** is a (p x p) square matrix such that: for $j \neq k$, $k_{jk} = 0$. **K** need not be the same for all i, $\Delta\boldsymbol{i}$, or **V**.

b) Issue responsiveness. For each issue, say m, there exists a voter, say s, and a change in her position on the m-th dimension, say Δs_m, such that if **V** and **V'** differ only in that s has changed position on m, then $d(\mathbf{V'})_m = d(\mathbf{V})_m + k\Delta s_m$, for some $k \neq 0$, where $d(\ .\ )_m$ is the decision under $\boldsymbol{d}(\ .\ )$ on the m-th issue.

A strictly quasi-concave, continuous, separable utility functions is symmetric if for each dimension, say m, i's loss on the m-th dimension from outcome x_m on that dimension is a strictly increasing function of $|x_m - i_m^*|$, where i_m^* is i's induced ideal point on the m-th dimension.

Theorem 1: Suppose that there are $p \geq 1$ issue dimensions and $n \geq 2$ voters with strictly quasi-concave, continuous, and separable utility functions. Then a resolute *ccf* satisfies anonymity, strategy-proofness and positive responsiveness if and only if it is a percentile method.

a) Necessity.

i) By resoluteness the *ccf* defines a decision with projection on to the m-th dimension d_m. (dropping reference to **V** for brevity). Number the voters in ascending order of the value of their positions on dimension m, arbitrarily numbering any voters who occupy the same positions. Let l_m and r_m the positions of the left-most (first) and right-most (n-th) voters respectively. Denote the set of voters with induced ideal point in $[l_m , d_m)$ by L_m. Denote the number of voters in L_m by $L_m^{\#}$. Suppose L_m is non-empty. Then there is contradiction between resoluteness, anonymity, responsiveness and strategy-proofness unless the decision remains d_m for all distributions of $L_m^{\#}$ positions on $(-\infty , d_m)$.

By anonymity d_m is only a function of the distribution of positions. Now consider any two distributions of positions of $L_m^{\#}$ voters over $(-\infty , d_m)$, say D1 and D2 where positions on other dimensions remain fixed. By resoluteness decisions are defined for D1 and for D2. Then compose a new distribution, S, where position i under S is the maximum of position i under D1 and position i under D2. By resoluteness a decision is defined for S. Now D1 can be transformed into S by changing one position at a time to its value under S. At each step the position either stays at the same value or increases. D2 can be transformed into S in a similar manner. In general there is a finite sequence of such moves from D1 to S, D1(2),….., D1(r), $r \leq n$.

Suppose the decision on the m-th dimension under D1 is d_m' and consider the shift from D1 to D1(1). Notice that by restricted responsiveness the decisions on other dimensions are fixed, because voter's position are assumed fixed on other dimensions. As voters' preferences are strictly quasi-concave and continuous, their induced preferences on the m-th dimension are single-peaked. One permissible form of utility function is, in addition, symmetric. Assume symmetry. Then, because preferences are separable, if the decision moves nearer a voter's induced ideal point on the m-th dimension than d_m', this unambiguously increases his payoff. Suppose the decision under D1(1), say $d_m(1)$, is different from that under D1. Let the leftmost voter's position in D1 be l_m and his position in D1(1) be $l_m(1)$. Then:

1. First consider the case in which and $d_m(1) > d_m'$. Now D1(1) is a possible sincere declaration under which citizens declare their ideal points. But then there is a contradiction with strategy-proofness: from distribution D1(1) the leftmost voter occupying position $l_m(1)$ could falsely declare her position as that of position 1 under distribution D1, l_m, and move the decision nearer her ideal point. So $d_m(1) = d_m'$.
2. Next consider the case in which and $d_m(1) < d_m'$ and $|l_m - d_m(1)| < |l_m - d_m'|$. Now D1 is a possible sincere declaration under which citizens declare their ideal points. But then there is a contradiction with strategy-proofness: from distribution D1 the voter occupying the leftmost position l_m could falsely declare her position as that of position 1 under distribution D1(1), $l_m(1)$, and move the decision nearer her ideal point. So $d_m(1) = d_m'$.
3. Finally consider the case in which and $d_m(1) < d_m'$ and $|l_m - d_m(1)| \geq |l_m - d_m'|$. Now D1(1) is a possible sincere declaration under which citizens declare their ideal points. But then there is a contradiction with strategy-proofness: from distribution D1(1) the voter occupying the leftmost position $l_m(1)$ could falsely declare her position as that of position 1 under distribution D1, l_m, and move the decision nearer her ideal point. So $d_m(1) = d_m'$.

Applying the same argument repeatedly to all intermediate shifts from D1(1) to S, the decision under S must be d_m', too. Now start with distribution D2. Using exactly the same arguments, whatever the decision is under D2, the decision under S is the same, or there is a contradiction. But this means that the decision under D2 must be the same as that under D1, d_m'. As the equivalence relation "gives rise to the same decision" holds between *any* pair of distributions of voters of $L_m^{\#}$ positions on $(-\infty, d_m)$, all such distributions belong to the same equivalence class. As, by assumption, the decision is d_m for one such distribution, it is the same for all of them Similarly if the set of members with declared positions to the right of d_m, R_m, is non-empty, the outcome must still be d_m for any distribution of $R_m^{\#}$ positions on the interval (d_m, ∞).
ii) The decision d_m does not change no matter how $L_m^{\#}$ voters are distributed on $(-\infty, d_m)$ or $R_m^{\#}$ voters on (d_m, ∞). But issue responsiveness requires that it is responsive to the position of at least one voter on the m-th issue. Any such voter can only be located at d_m. Thus on the m-th dimension the decision is the position of the $(L_m^{\#}+1)^{th}$ voter. As a similar argument holds for each dimension, the *ccf* is a percentile method.

b) *Sufficiency.*

i) Percentile methods are resolute as this only requires that positions on each dimension can be indexed from 1 to n.

ii) Percentile methods satisfy anonymity, because they are only a function of positions.

iii) Percentile methods satisfy responsiveness. They satisfy restricted responsiveness as the decision on an issue is a function only of positions on that issue. On the m-th dimension the decision corresponds to the position of at least one voter. Suppose voter i is the lowest-indexed member of the set P_m who are positioned at the decision. Then the decision responds to movements of her position to the left when she shifts in the open interval between the furthest right member of L_m (or $-\infty$ if L_m is empty) and d_m; so the decision satisfies issue responsiveness.

v) With separable preferences percentile methods are strategy-proof. Suppose the declaration **V** is sincere. Because preferences are separable, we can consider changes in strategy on dimensions one at a time. Consider dimension m. Voters in L_m may be able to change the decision to the right by moving to the right of the voter at d_m, but this would be to move it further away from their induced ideal point, which would decrease their payoff. Similarly voters in R_m cannot strictly increase their payoff by shifting positions. Individual members of P_m at d_m may be able to change the outcome either to the right or to the left, depending on how many voters are at this position; but they have no incentive to do so since d_m corresponds to their induced ideal-point. □

Theorem 2: Suppose that there $n \geq 2$ voters with strictly quasi-concave, continuous, separable, symmetric cardinal utility functions whose positions on each of $p \geq 1$ issue dimensions are bounded. Then from behind the veil of ignorance, among the percentile methods the issue-by-issue median of positions is the percentile method that minimises expected losses.

By separability, expected utility can be maximised one dimension at a time. To simplify notation drop the reference to particular dimensions.

From behind the veil of ignorance voters assume they will take up position at their ideal point if the *ccf* is a percentile method, consistent with this strategy forming part of a Nash equilibrium. As positions are bounded and utility functions are defined only up to a positive affine transformation, we might as well think of ideal points as lying in the interval [0 , 1].

From behind the veil of ignorance ideal points are random variables X_1, X_2, …., X_n, each independently drawn from the uniform distribution on [0 , 1], because any voter is equally likely to occupy any position on the interval and positions are independent.

Arrange the random position variables in ascending order, writing them as $X_{1:n} \leq X_{2::n} \leq$ ……..$\leq X_{n::n}$, where $X_{i::n}$ is the i-th order statistic out of n. Suppose the percentile method selects the d-th position from the left. Then from behind the veil of ignorance the decision is the random variable $X_{d::n}$.

Consider some realisation ρ of the n order statistics, $\rho = \{x_1, x_2, \ldots x_{n-1}, x_n\}$ – a possible conjuncture as seen from behind the veil of ignorance. Suppose in this realisation the decision is x_d and voter i's ideal point is x_i. Then i's loss is can be represented as $-l(d_i)$, where $d_i = |x_i - x_d|$, $dl/dd_i > 0$, as his utility function is single peaked and symmetric. Then if we set

$$\Lambda(\rho)_d = \sum_{j \leq n} -l(d_j),$$

i's expected loss from decision x_d in realisation ρ is $\Lambda(\rho)_d /n$, as from behind the veil of ignorance he is equally likely to take any of the n positions in the conjuncture concerned.

Suppose, first, that n is odd, so the median of positions m = (n+1)/2.

1) Now consider a realisations *conditional on* $X_{n:n} = x$. Thus in such a realization r, $x_1 \leq x_2$ …..$\leq x_{m-1} \leq x_m \leq$ ……$\leq x_{n-1} \leq x$. Consider the expected restricted loss function, defined over losses for the (n-1) unconstrained draws,

$$\Lambda(r)_{d,n-1} = \sum_{j \leq n-1} -l(d_j)$$

Now suppose in some realization r that $\Lambda(r)_{m-1,n-1} > \Lambda(r)_{m,n-1}$. Then there exists a realization r' such that $\Lambda(r')_{m,n-1} > \Lambda(r')_{m-1,n-1}$. Specifically r' is the mirror image of r, so that $x_j' = x - x_{n-j}$, $1 \leq j \leq n-1$; so, as l(.) is a symmetric function about its minimum, $\Lambda(r)_{j,n-1} = \Lambda(r')_{n-j,n-1}$. But then $\Lambda(r')_{m,n-1} = \Lambda(r)_{m-1,n-1}$ and $\Lambda(r')_{m-1,n-1} = \Lambda(r)_{m,n-1}$.

2) Conditional on the fixed value of the n-th order statistic, x, the joint *pdf* for the (n-1) other order statistics is $(n-1)!/x^{n-1}$ (Balakrishnan and Clifford Cohen, 1991, 15); so pairs of mirror-image realisations like r and r' each have the same density. The class of realisations

contingent on x consists of those where the decisions (m-1) and m have the same losses and those that do not; but as we have seen the later subset can be divided into mirror image pairs such that the expected restricted loss from decision m in one realization is the expected restricted loss from decision m-1 in the other. Thus allowing for realisations where losses are the same and integrating over such mirror image pairs and all possible values of x, $E(\Lambda_{m-1,n-1}) = E(\Lambda_{m,n-1})$, where the first term denotes the expected loss over the first (n-1) order statistics when the decision is the (m-1)th order statistic, etc.

3) $|x_{:n} - x_m| \leq |x_{:n} - x_{m-1}|$ in any realization and there are realisations with strictly positive density such that $|x_n - x_m| < |x_n - x_{m-1}|$. Hence $E(-l(m_{:n})) > E(-l(m\text{-}1_{:n}))$. Generally for decision d, $E(\Lambda_{d,n}) = E(\Lambda_{d,n-1}) + E(-l(d_{:n}))$. So, as $E(\Lambda_{m-1,n-1}) = E(\Lambda_{m,n-1})$, $E(\Lambda_{m,n}) > E(\Lambda_{m-1,n})$.

4) Suppose that $m-2 \geq 1$. Then fix $X_{n,n} = x$, $X_{n,n-1} = x'$ and $X_{n,n-2} = x''$, $x \geq x' \geq x''$. Then using similar arguments to those in 1) and 2) above, $E(\Lambda_{m-2,n-3}) = E(\Lambda_{m-1,n-3})$.
$E(\Lambda_{d,n}) = E(\Lambda_{d,n-3}) + E(-l(d_{:n}) + E(-l(d_{n-1})) + E(-l(d_{n-2})$, so using similar arguments to 3) , $E(\Lambda_{m-1,n}) > E(\Lambda_{m-2,n})$.

5) Using similar arguments to 4), $E(\Lambda_{m-l,n}) > E(\Lambda_{m-1-1,n})$, for all $l \leq m-2$, hence $E(\Lambda_{m,n}) > E(\Lambda_{m-l,n})$ for all $l \leq m-2$.

6) Using symmetrical arguments to 1) through 5) applied to decisions on the right of the median, $E(\Lambda_{m,n}) > E(\Lambda_{m+l,n})$, for all $l \leq n-m$.

7) Hence $E(\Lambda_{m,n}) > E(\Lambda_{d',n})$ for any $d' \neq m$. I.e. for $n \geq 3$ and n odd, i's expected payoff is greatest when $d = m = (n+1)/2$, as asserted.

Now suppose that n is even and $n \geq 4$. Here the left median is $m_l = n/2$ and the right median is $m_r = n/2 + 1$.

8) By similar arguments to 1) through 5), $E(\Lambda_{ml,n}) > E(\Lambda_{ml-q,n})$, for $q \leq m_l-1$.

9) Then using a similar argument to 6) above $E(\Lambda_{mr,n}) > E(\Lambda_{mr+q,n})$, $q \leq n - m_r$.

10) Using similar arguments to 1) and 2) above, $E(\Lambda_{ml,n}) = E(\Lambda_{mr,n})$; hence the left and right medians are the decisions with the lowest expected losses for even n, $n \geq 4$ as asserted.

11) For n=2 the only percentile methods are to choose either the right or the left median; and trivially these have equal expected losses. □

References.

Balakrishnan, Narayanaswamy and Alonzo Clifford-Cohen (1991) *Order Statistics and Inference: Estimation Methods* (Academic Press, New York).